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DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR ELLI--ETC(U)

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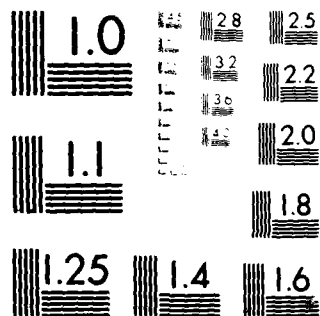
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DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR
ELLIPTICALLY CONTOURED DISTRIBUTIONS AND THEIR APPLICATIONS

BY

T. W. ANDERSON and KAI-TAI FANG

TECHNICAL REPORT NO. 53
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THEODORE W. ANDERSON, PROJECT DIRECTOR

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DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR
ELLIPTICALLY CONTOURED DISTRIBUTIONS AND THEIR APPLICATIONS

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1. Introduction.

If the characteristic function of an n -dimensional random vector \underline{x} has the form $e^{it'\underline{\mu}}\phi(t'\underline{\Sigma}t)$, where $\underline{\mu}:n\times 1$, $\underline{\Sigma}:n\times n$, and $\underline{\Sigma} \geq 0$, we say that \underline{x} is distributed according to an elliptically contoured distribution with parameters $\underline{\mu}$, $\underline{\Sigma}$ and ϕ , and we write $\underline{x} \sim EC_n(\underline{\mu}, \underline{\Sigma}, \phi)$ (cf. Cambanis, Huang, and Simons (1981)). In particular, when $\underline{\mu} = 0$ and $\underline{\Sigma} = \underline{I}_n$ (the identity matrix), $EC_n(0, \underline{I}_n, \phi)$ is called a spherical distribution (cf. Kelker (1970)).

The class of the elliptically contoured distributions contains such distributions as the multivariate normal distribution, the multivariate t -distribution, the multivariate Cauchy distribution, the multivariate Laplace distribution and the multivariate uniform distribution on the sphere in \mathbb{R}^n or in the sphere in \mathbb{R}^n .

The theory of elliptically contoured distributions has been discussed by Schoenberg (1938), Lord (1954), Kelker (1970), Das Gupta et al. (1972),

Devlin, Gnanadesikan and Kettenring (1976), Kariya and Eaton (1977), Muirhead (1980) and Cambanis, Huang, and Simons (1981). The following basic properties obtained by the above authors are needed in this paper.

(1) Let $\underline{u}^{(n)}$ denote a random vector which is uniformly distributed on the unit sphere in \mathbb{R}^n and $\Omega_n(\|\underline{t}\|^2)$ denote its characteristic function. If Φ_n , $n \geq 1$, is the class of all functions $\phi: [0, \infty) \rightarrow \mathbb{R}$ such that $\phi(\|\underline{t}\|^2)$ is a characteristic function, then $\phi \in \Phi_n$ if and only if

$$(1.1) \quad \phi(t) = \int_0^\infty \Omega_n(r^2 t) dF(r), \quad t \geq 0,$$

for some distribution function F on $[0, \infty)$ (cf. Schoenberg (1938)).

(2) $\underline{x} \sim EC_n(\underline{\mu}, \underline{\Sigma}, \phi)$ with rank $rk(\underline{\Sigma}) = k$ if and only if

$$(1.2) \quad \underline{x} \stackrel{d}{=} \underline{\mu} + R \underline{A}' \underline{u}^{(k)}$$

where $R \geq 0$ is independent of $\underline{u}^{(k)}$, $\underline{\Sigma} = \underline{A}'\underline{A}$ is a factorization of $\underline{\Sigma}$ (i.e., \underline{A} is a $k \times n$ matrix and $rk(\underline{A}) = k$) and the distribution function $F(x)$ of R is related to ϕ as in (1.1) with k substituted for n . Here, $\underline{x} \stackrel{d}{=} \underline{y}$ denotes that the random vectors \underline{x} and \underline{y} are identically distributed (cf. Cambanis, Huang, and Simons (1981)). In the next section we discuss the operation " $\stackrel{d}{=}$ " in detail.

(3) If $\underline{x} \sim N_n(\underline{\mu}, \underline{\Sigma})$, then $\underline{x} \sim EC_n(\underline{\mu}, \underline{\Sigma}, \phi)$ with $\phi(t) = \exp(-t/2)$ and $R^2 \sim \chi_n^2$, where χ_n^2 denotes the chi-squared distribution with n degrees of freedom.

If \underline{A} is an $n \times n$ symmetric matrix, what is the distribution of $\underline{x}'\underline{A}\underline{x}$? Kelker (1970) obtained the distribution of $\underline{x}'\underline{A}\underline{x}/\underline{x}'\underline{x}$ in the case

of $\underline{x} \sim EC_n(0, \underline{I}_n, \phi)$ and the distribution of $\underline{x}'\underline{A}\underline{x}$ under the condition that \underline{x} has finite fourth moments and \underline{x} has a density. Kariya and Eaton (1977) gave the distributions of $\underline{a}'\underline{x}/\|\underline{x}\|$ and $\underline{x}'\underline{A}\underline{x}/\|\underline{x}\|^2$, where $\|\underline{x}\|^2 = \underline{x}'\underline{x}$. There is a rich bibliography on the distribution of quadratic forms in normal population. Recently, Anderson and Styan (1980) reviewed various extensions of Cochran's Theorem in a bibliographic and historical setting. Our purpose in this paper is to extend Cochran's Theorem to elliptically contoured distributions in various aspects. The main results are in Sections 4 and 5. In Section 3 we list some basic properties of Dirichlet distributions that are needed later. Some applications are given in Section 6.

Throughout the paper $N_n(\underline{\mu}, \underline{\Sigma})$ denotes the n -dimensional normal distribution with mean $\underline{\mu}$ and covariance matrix $\underline{\Sigma}$, χ_k^2 denotes the chi-squared distribution with k degrees of freedom, $F(k, \ell)$ denotes the F -distribution with k and ℓ degrees of freedom, t_n denotes the t -distribution with n degrees of freedom, \underline{I}_n denotes the $n \times n$ identity matrix, \underline{e}_n denotes the $n \times 1$ vector with elements 1, $\text{rk}(\underline{A})$ denotes the rank of the matrix \underline{A} , and \underline{A}^- denotes a generalized inverse.

2. The Operation " $\stackrel{d}{=}$ ".

If random vectors \underline{x} and \underline{y} have the same distribution, we denote that fact by $\underline{x} \stackrel{d}{=} \underline{y}$.

(1) Assume \underline{x} , \underline{y} , \underline{z} and \underline{w} are random vectors, \underline{x} and \underline{z} are independent, \underline{y} and \underline{w} are independent, and $\underline{x} \stackrel{d}{=} \underline{y}$. Then $\underline{z} \stackrel{d}{=} \underline{w}$ if and only if

$$(2.1) \quad \underline{x} + \underline{z} \stackrel{d}{=} \underline{y} + \underline{w}.$$

In particular, if \underline{z} is a nonrandom vector, then $\underline{x} = \underline{y}$ implies $\underline{x} + \underline{z} \stackrel{d}{=} \underline{y} + \underline{z}$.

(2) Assume that $\underline{x} \stackrel{d}{=} \underline{y}$ and $f_j(\cdot)$, $j = 1, \dots, m$, are Borel functions, then

$$(2.2) \quad \begin{pmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \\ \vdots \\ f_m(\underline{x}) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} f_1(\underline{y}) \\ f_2(\underline{y}) \\ \vdots \\ f_m(\underline{y}) \end{pmatrix}.$$

For instance, we have

$$(2.3) \quad \underline{x}' \underline{A} \underline{x} \stackrel{d}{=} \underline{y}' \underline{A} \underline{y}, \quad \underline{x}' \underline{x} \stackrel{d}{=} \underline{y}' \underline{y},$$

$$(2.4) \quad \begin{pmatrix} \underline{x}' \underline{A}_1 \underline{x} \\ \vdots \\ \underline{x}' \underline{A}_m \underline{x} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \underline{y}' \underline{A}_1 \underline{y} \\ \vdots \\ \underline{y}' \underline{A}_m \underline{y} \end{pmatrix}.$$

(3) The following fact is important in this paper.

Lemma 1. Assume that \underline{x} and \underline{y} are $n \times 1$ random vectors, z is a random variable and is independent of \underline{x} and \underline{y} , respectively. If

$$(2.5) \quad P(z > 0) = 1$$

then $\underline{x} \stackrel{d}{=} \underline{y}$ if and only if $z \underline{x} \stackrel{d}{=} z \underline{y}$.

Proof. Firstly, we prove that Lemma 1 holds if

$$(2.6) \quad P(x_i > 0) = P(y_i > 0) = 1, \quad i = 1, \dots, n.$$

By using (2.2) we have

$$\begin{aligned} \underline{zx} \stackrel{d}{=} \underline{zy} &\Leftrightarrow \begin{pmatrix} \ln zx_1 \\ \vdots \\ \ln zx_n \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \ln zy_1 \\ \vdots \\ \ln zy_n \end{pmatrix} \Leftrightarrow \ln z\varepsilon_n + \begin{pmatrix} \ln x_1 \\ \vdots \\ \ln x_n \end{pmatrix} \stackrel{d}{=} \ln z\varepsilon_n + \begin{pmatrix} \ln y_1 \\ \vdots \\ \ln y_n \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \ln x_1 \\ \vdots \\ \ln x_n \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \ln y_1 \\ \vdots \\ \ln y_n \end{pmatrix} \Leftrightarrow \underline{x} \stackrel{d}{=} \underline{y} . \end{aligned}$$

Secondly, we consider \underline{x} and \underline{y} to be arbitrary random vectors. It is enough to prove the lemma in the case of \underline{x} and \underline{y} being scalar random variables because

$$\begin{aligned} \underline{x} \stackrel{d}{=} \underline{y} &\Leftrightarrow \underline{a}'\underline{x} \stackrel{d}{=} \underline{a}'\underline{y}, \quad \forall \underline{a} \in \mathbb{R}_n \Leftrightarrow \underline{za}'\underline{x} \stackrel{d}{=} \underline{za}'\underline{y}, \quad \forall \underline{a} \in \mathbb{R}_n \\ &\Leftrightarrow \underline{a}'(\underline{zx}) \stackrel{d}{=} \underline{a}'(\underline{zy}), \quad \forall \underline{a} \in \mathbb{R}_n \Leftrightarrow \underline{zx} \stackrel{d}{=} \underline{zy} \end{aligned}$$

if we have proved the lemma in the above case. Let

$$f_+(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad f_-(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and $x^+ = f_+(x)$, $x^- = f_-(x)$; $y^+ = f_+(y)$ and $y^- = f_-(y)$. It is easy to verify that

$$\underline{x} \stackrel{d}{=} \underline{y} \Leftrightarrow \underline{x}^+ \stackrel{d}{=} \underline{y}^+, \quad \underline{x}^- \stackrel{d}{=} \underline{y}^-, \quad \text{and} \quad p(x=0) = p(y=0) \Leftrightarrow \underline{x}^+ \stackrel{d}{=} \underline{y}^+ \quad \text{and} \quad \underline{x}^- \stackrel{d}{=} \underline{y}^-.$$

If we can prove that $\underline{zx}^+ \stackrel{d}{=} \underline{zy}^+ \Leftrightarrow \underline{x}^+ \stackrel{d}{=} \underline{y}^+$ and $\underline{zx}^- \stackrel{d}{=} \underline{zy}^- \Leftrightarrow \underline{x}^- \stackrel{d}{=} \underline{y}^-$, then the assertion follows from

$$zx \stackrel{d}{=} zy \Leftrightarrow (zx)^+ \stackrel{d}{=} (zy)^+ \text{ and } (zx)^- \stackrel{d}{=} (zy)^- \Leftrightarrow$$

$$zx^+ \stackrel{d}{=} zy^+ \text{ and } zx^- \stackrel{d}{=} zy^- \Leftrightarrow x^+ \stackrel{d}{=} y^+ \text{ and } x^- \stackrel{d}{=} y^- \Leftrightarrow x \stackrel{d}{=} y.$$

Now we prove that $zx^+ \stackrel{d}{=} zy^+ \Leftrightarrow x^+ \stackrel{d}{=} y^+$. Assume $zx^+ \stackrel{d}{=} zy^+$, then

$$p(x>0) = p(x^+>0) = p(zx^+>0) = p(zy^+>0) = p(y^+>0) = p(y>0)$$

and

$$\begin{aligned} p(x>0)E(e^{itzx}|x>0) &= E(e^{itzx^+}) - p(x\leq 0) = E(e^{itzy^+}) - p(y\leq 0) \\ &= E(e^{itzy}|y>0) p(y>0). \end{aligned}$$

From the first part of the proof we have $E(e^{itzx}|x>0) = E(e^{itzy}|y>0)$, i.e., $x^+ \stackrel{d}{=} y^+$. The " \Leftarrow " part follows by the same technique. Similarly we have $zx^- \stackrel{d}{=} zy^- \Leftrightarrow x^- \stackrel{d}{=} y^-$. Q.E.D.

3. The Dirichlet Distribution.

If $y = (y_1, \dots, y_m)'$ is a random vector with $\sum_1^m y_i = 1$ and $(y_1, \dots, y_{m-1})'$ has the density with $\alpha_i > 0$, $i = 1, \dots, m$,

$$(3.1) \quad p_m(t_1, \dots, t_{m-1}) = \frac{\Gamma\left(\sum_{i=1}^m \alpha_i\right)}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^{m-1} t_i^{\alpha_i-1} \left(1 - \sum_{i=1}^{m-1} t_i\right)^{\alpha_m-1}$$

$$\text{if } t_i \geq 0, \quad i = 1, \dots, m-1, \quad \sum_{i=1}^{m-1} t_i < 1,$$

$$= 0, \quad \text{otherwise.}$$

We say that y is distributed according to a Dirichlet distribution and

denote it as $(y_1, \dots, y_{m-1}) \sim D_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m)$. When $m=2$ $D_2(d_1; d_2)$ reduces to the Beta distribution, $B(\alpha_1, \alpha_2)$.

It is a well-known fact that if $\underline{x} \sim N_n(0, I_n)$ and is partitioned into m parts $\underline{x}_{(1)}, \dots, \underline{x}_{(m)}$ with $\alpha_1, \dots, \alpha_m$ components of \underline{x} , respectively, i.e.,

$$(3.2) \quad \underline{x} = \begin{pmatrix} \underline{x}_{(1)} \\ \vdots \\ \underline{x}_{(m)} \end{pmatrix},$$

then the joint density of $\left(\frac{\underline{x}'_{(1)} \underline{x}_{(1)}}{\|\underline{x}\|^2}, \dots, \frac{\underline{x}'_{(m-1)} \underline{x}_{(m-1)}}{\|\underline{x}\|^2} \right)$ is $D_m\left(\frac{\alpha_1}{2}, \dots, \frac{\alpha_{m-1}}{2}; \frac{\alpha_m}{2}\right)$.

In particular, if $\alpha_1 = \dots = \alpha_k = 1$, $\alpha_{k+1} = n-k$, $m = k+1$, then

$\left(\frac{x_1^2}{\|\underline{x}\|^2}, \dots, \frac{x_k^2}{\|\underline{x}\|^2} \right)$ has $D_{k+1}\left(\frac{1}{2}, \dots, \frac{1}{2}; \frac{n-k}{2}\right)$ as its density. It can be veri-

fied that the density of $\left(\frac{|x_1|}{\|\underline{x}\|}, \dots, \frac{|x_k|}{\|\underline{x}\|} \right)$ is

$$(3.3) \quad \frac{\Gamma(\frac{n}{2}) 2^k}{\Gamma(\frac{n-k}{2}) \pi^{k/2}} \left(1 - \sum_{i=1}^k x_i^2 \right)^{\frac{n-k}{2} - 1} \quad \text{if } x_i \geq 0, \quad i=1, \dots, k; \quad \sum_{i=1}^k x_i^2 < 1,$$

and the density of $\left(\frac{x_1}{\|\underline{x}\|}, \dots, \frac{x_k}{\|\underline{x}\|} \right)$ is

$$(3.4) \quad g_n(x_1, \dots, x_k) \equiv \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2}) \pi^{k/2}} \left(1 - \sum_{i=1}^k x_i^2 \right)^{\frac{n-k}{2} - 1}, \quad \text{if } \sum_{i=1}^k x_i^2 < 1.$$

If $\underline{x} \sim E C_n(0, I_n \phi)$, then (cf. (1.2))

$$(3.5) \quad \underline{x} \stackrel{d}{=} R \underline{u}^{(n)},$$

where R and $\underline{u}^{(n)}$ are independent. From (2.3)

$$(3.6) \quad \|\underline{x}\|^2 = \underline{x}'\underline{x} \stackrel{d}{=} R^2 \underline{u}^{(n)'} \underline{u}^{(n)} = R^2 \quad \text{or} \quad \|\underline{x}\| \stackrel{d}{=} R.$$

If $p(\underline{x}=0) = 0$ (or $p(R=0)=0$), we have (cf. Cambanis, Huang, and Simons, (1981), Lemma 1)

$$(3.7) \quad \underline{x}/\|\underline{x}\| \stackrel{d}{=} \underline{u}^{(n)},$$

and it is independent of $\|\underline{x}\| \stackrel{d}{=} R$. Note that (3.7) holds for all of the class of spherical distributions with $p(\underline{x}=0) = 0$. Thus we can assume $\underline{x} \sim N_n(0, I_n)$ to obtain the distribution of ratios. We will use the property many times. Denote $\underline{u}^{(n)} = (u_1, \dots, u_n)'$. We immediately obtain that

- (i) the density of u_{i_1}, \dots, u_{i_k} , $1 \leq i_1 < i_2 < \dots < i_k \leq n$ is $g_n(x_1, \dots, x_k)$, (cf. (3.4));
- (ii) the density of $|u_{i_1}|, \dots, |u_{i_k}|$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$, is (3.3);
- (iii) if $\underline{u}^{(n)}$ is partitioned into m parts as the same as (3.2), then the density of $(\|\underline{u}_{(1)}\|^2, \dots, \|\underline{u}_{(m-1)}\|^2)$ is $D_m(\frac{1}{2} \alpha_1, \dots, \frac{1}{2} \alpha_{m-1}; \frac{1}{2} \alpha_m)$.
- (iv) Let $\mu_{r_1, \dots, r_m} = E[\prod_{i=1}^m u_i^{r_i}]$, then $\mu_{r_1, \dots, r_m} = 0$ if some r_i are odd. If all of r_1, \dots, r_m are even, μ_{r_1, \dots, r_m} can be obtained as moments of the Dirichlet distribution (cf. Johnson and Kotz (1972)), i.e.

$$(3.8) \quad \mu_{r_1, \dots, r_m} = \frac{\prod_{j=1}^m \left[\frac{1}{2} \right]^{r_j}}{\left[\frac{n}{2} \right]^{1 + \sum_{j=1}^m r_j}},$$

where $x^{[k]} = x(x+1) \dots (x+k-1)$. In particular, $E \underline{u}^{(n)} = \underline{0}$, $E \underline{u}^{(n)} \underline{u}^{(n)'} = \frac{1}{n} I_n$.

There is a close relationship between the chi-squared distribution and Dirichlet distribution from the above discussion. The following are further results.

Lemma 2. Assume that $(z_1, \dots, z_m) \sim D_{m+1}(\frac{1}{2} \alpha_1, \dots, \frac{1}{2} \alpha_m; \frac{1}{2} \alpha_{m+1})$ with $\alpha_i > 0, i = 1, \dots, m+1$, and $\sum_{i=1}^{m+1} \alpha_i = n$ and y_0, y_1, \dots, y_{m+1} are distributed as chi-squared distributions with degrees of freedom $n, \alpha_1, \dots, \alpha_{m+1}$, respectively; then

$$(3.9) \quad \phi_{\ell_n z_1, \dots, \ell_n z_m}(t_1, \dots, t_m) = \frac{\prod_{j=1}^m \phi_{\ell_n y_j}(t_j)}{\phi_{\ell_n y_0}(\sum_{j=1}^m t_j)} \\ = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2} + i(t_1 + \dots + t_m))} \prod_{j=1}^m \frac{\Gamma(\frac{\alpha_j}{2} + i t_j)}{\Gamma(\frac{\alpha_j}{2})},$$

where $\phi(\cdot)$ denotes the characteristic function and $i = \sqrt{-1}$.

Proof. Let $\underline{x} \sim N_n(0, I_n)$ and $\underline{x}_{(1)}, \dots, \underline{x}_{(m+1)}$ have the similar sense as (3.2). Then $\|\underline{x}_{(j)}\|^2 \stackrel{d}{=} y_j, j = 1, \dots, m+1$, and $\|\underline{x}\|^2 \stackrel{d}{=} y_0$. We have

$$\begin{bmatrix} \|\underline{x}_{(1)}\|^2 \\ \vdots \\ \|\underline{x}_{(m)}\|^2 \end{bmatrix} \stackrel{d}{=} \begin{bmatrix} \|\underline{x}_{(1)}\|^2 / \|\underline{x}\|^2 \\ \vdots \\ \|\underline{x}_{(m)}\|^2 / \|\underline{x}\|^2 \end{bmatrix} \cdot \|\underline{x}\|^2,$$

and

$$\begin{bmatrix} \ell_n \|\underline{x}_{(1)}\|^2 \\ \vdots \\ \ell_n \|\underline{x}_{(m)}\|^2 \end{bmatrix} \stackrel{d}{=} \ell_n \|\underline{x}\|^2 \underline{e}_n + \begin{bmatrix} \ell_n \|\underline{x}_{(1)}\|^2 / \|\underline{x}\|^2 \\ \vdots \\ \ell_n \|\underline{x}_{(m)}\|^2 / \|\underline{x}\|^2 \end{bmatrix}.$$

The first part of (3.9) follows from the fact that $\|\underline{x}\|$ and $\underline{x}/\|\underline{x}\|$ are independent and $(\|\underline{x}_{(1)}\|^2/\|\underline{x}\|^2, \dots, \|\underline{x}_{(m)}\|^2/\|\underline{x}\|^2) \stackrel{d}{=} (z_1, \dots, z_m)$. Note (cf. Press (1969))

$$E(e^{it \ell_n y_j}) = \frac{2^{it} \Gamma(\frac{\alpha_j}{2} + it)}{\Gamma(\frac{\alpha_j}{2})};$$

the second part of (3.9) follows. Q.E.D.

In particular, if $z \sim B(\frac{k}{2}, \frac{n-k}{2})$, $y_0 \sim \chi_n^2$ and $y_1 \sim \chi_k^2$, then

$$(3.10) \quad \phi_{\ell_n z}(t) = \frac{\phi_{\ell_n y_1}(t)}{\phi_{\ell_n y_0}(t)} = \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{k}{2} + it)}{\Gamma(\frac{k}{2}) \Gamma(\frac{n}{2} + it)} = \frac{B(\frac{k}{2} + it, \frac{n-k}{2})}{B(\frac{k}{2}, \frac{n-k}{2})},$$

where $B(a, b)$ is Beta function.

Lemma 3. Assume that $\underline{x} \sim N_n(0, I_n)$, $(z_1, \dots, z_{n-1}) \sim D_n(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$, $z_n = 1 - \sum_{i=1}^{n-1} z_i$, y_1, \dots, y_n are i.i.d., $y_i \sim \chi_1^2$, $i = 1, \dots, n$, and A_1, \dots, A_m are $n \times n$ symmetric matrices. Let $g_i(t_1, \dots, t_n)$, $i = 1, \dots, m$, be linear functions of t_1, \dots, t_n , then

$$(3.11) \quad \begin{pmatrix} \underline{x}' A_1 \underline{x} / \|\underline{x}\|^2 \\ \vdots \\ \underline{x}' A_m \underline{x} / \|\underline{x}\|^2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g_1(z_1, \dots, z_n) \\ \vdots \\ g_m(z_1, \dots, z_n) \end{pmatrix} \approx \begin{pmatrix} \underline{x}' A_1 \underline{x} \\ \vdots \\ \underline{x}' A_m \underline{x} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g_1(y_1, \dots, y_n) \\ \vdots \\ g_m(y_1, \dots, y_n) \end{pmatrix}.$$

Proof. We firstly prove the implication towards the left. If the right side of (3.11) holds then

$$\|x\|^2 \begin{pmatrix} x'A_1x/\|x\|^2 \\ \vdots \\ x'A_mx/\|x\|^2 \end{pmatrix} = \begin{pmatrix} x'A_1x \\ \vdots \\ x'A_mx \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g_1(y_1, \dots, y_n) \\ \vdots \\ g_m(y_1, \dots, y_n) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g_1(x_1^2, \dots, x_n^2) \\ \vdots \\ g_m(x_1^2, \dots, x_n^2) \end{pmatrix} = \|x\|^2 \begin{pmatrix} g_1(\frac{x_1^2}{\|x\|^2}, \dots, \frac{x_n^2}{\|x\|^2}) \\ \vdots \\ g_m(\frac{x_1^2}{\|x\|^2}, \dots, \frac{x_n^2}{\|x\|^2}) \end{pmatrix}$$

and from Lemma 1

$$\begin{pmatrix} x'A_1x/\|x\|^2 \\ \vdots \\ x'A_mx/\|x\|^2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g_1(\frac{x_1^2}{\|x\|^2}, \dots, \frac{x_n^2}{\|x\|^2}) \\ \vdots \\ g_m(\frac{x_1^2}{\|x\|^2}, \dots, \frac{x_n^2}{\|x\|^2}) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g_1(z_1, \dots, z_n) \\ \vdots \\ g_m(z_1, \dots, z_n) \end{pmatrix}.$$

Here we use the fact again that $\|x\|^2$ and $x/\|x\|$ are independent. The proof in the other direction is similar. Q.E.D.

By Lemma 3, we can change the theory of distributions of quadratic forms from chi-square distributions to Dirichlet distributions.

Corollary 1. Assume that $x \sim N_n(0, I)$ and A is an $n \times n$ symmetric matrix, then $x'Ax/\|x\|^2 \sim B(\frac{k}{2}, \frac{n-k}{2})$ if and only if $A^2 = A$ and $\text{rk}(A) = k$.

Proof. By Cochran's Theorem, $A^2 = A$ and $\text{rk}(A) = k$ if and only if $x'Ax \sim \chi_k^2$ or $x'Ax \stackrel{d}{=} \sum_1^k y_i$, hence if and only if $x'Ax/\|x\|^2 = \sum_1^k z_i \sim B(\frac{k}{2}, \frac{n-k}{2})$ from Lemma 3 with $m = 1$ and $g_1(t_1, \dots, t_n) = t_1 + \dots + t_k$. Q.E.D.

Corollary 2. Assume that $\underline{x} \sim N_n(0, \underline{I}_n)$, \underline{A} and \underline{B} are $n \times n$ symmetric matrices, then z_1, \dots, z_{n-1} has the Dirichlet distribution $D_n(\frac{1}{2}, \dots, \frac{1}{2}; \frac{1}{2})$ and $\sum_1^n z_i = 1$, then

$$(3.12) \quad \left(\frac{\underline{x}' \underline{A} \underline{x}}{\|\underline{x}\|^2}, \frac{\underline{x}' \underline{B} \underline{x}}{\|\underline{x}\|^2} \right) \stackrel{d}{=} \left(\sum_1^k \lambda_i z_i, \sum_{k+1}^{k+m} v_j z_j \right), \quad k+m \leq n,$$

if and only if $\underline{AB} = 0$, where λ_i and v_j are nonzero real numbers.

Proof. By Craig's Theorem (cf. Anderson and Styán (1980)) $\underline{AB} = 0$ if and only if $\underline{x}' \underline{A} \underline{x}$ and $\underline{x}' \underline{B} \underline{x}$ are independent or if and only if

$$(\underline{x}' \underline{A} \underline{x}, \underline{x}' \underline{B} \underline{x}) \stackrel{d}{=} \left(\sum_1^k \lambda_i y_i, \sum_1^m v_j y_{j+k} \right).$$

where $\lambda_1, \dots, \lambda_k$ and v_1, \dots, v_m are the nonzero eigenvalues of \underline{A} and \underline{B} , respectively. Thus the corollary follows by Lemma 3. Q.E.D.

Corollary 3. Let \underline{x} , \underline{A} and \underline{B} be defined as in Corollary 2, then $(\underline{x}' \underline{A} \underline{x} / \|\underline{x}\|^2, \underline{x}' \underline{B} \underline{x} / \|\underline{x}\|^2) \sim D_3(\frac{k}{2}, \frac{m}{2}; \frac{n-k-m}{2})$, with $k > 0$, $m > 0$, $k+m \leq n$, if and only if $\underline{AB} = 0$, $\underline{A}^2 = \underline{A}$, $\underline{B}^2 = \underline{B}$, $\text{rk}(\underline{A}) = k$ and $\text{rk}(\underline{B}) = m$.

Proof. The corollary follows from Corollary 2 and $(\sum_1^k z_i, \sum_{k+1}^{k+m} z_i) \sim D_3(\frac{k}{2}, \frac{m}{2}; \frac{n-k-m}{2})$. Q.E.D.

4. Distributions of Quadratic Forms and Cochran's Theorem.

In this section we want to extend distributions of quadratic forms and Cochran's Theorem to the case of elliptically contoured distributions with mean zero. Firstly, we need to generalize the Dirichlet distribution and the Beta distribution.

If a random vector $(z_1, \dots, z_m)'$ satisfies $(z_1, \dots, z_m) \stackrel{d}{=} R^2(u_1, \dots, u_m)$, where $R \sim F(x)$, R is independent of (u_1, \dots, u_{m-1}) , $\sum_{i=1}^m u_i = 1$, and $(u_1, \dots, u_{m-1}) \sim D_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m)$, then we write $(z_1, \dots, z_{m-1}) \sim G_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m; \phi)$ and $(z_1, \dots, z_m) \sim G_m(\alpha_1, \dots, \alpha_{m-1}, \alpha_m; \phi)$, where ϕ is related to $F(x)$ as in (1.1) with $n = 2(\alpha_1 + \dots + \alpha_m)$.

It is easy to show that the density of $G_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m; \phi)$ is

$$(4.1) \quad \frac{\Gamma(\frac{n}{2})}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^{m-1} z_i^{\alpha_i-1} \int_{\sqrt{\sum_{i=1}^{m-1} z_i}}^{\infty} r^{-(n-2)} \left(r^2 - \sum_{i=1}^{m-1} z_i \right)^{\alpha_m-1} dF(r), \text{ if } z_i \geq 0.$$

Further, if R has a density $f(r)$, then the joint density of u_1, \dots, u_{m-1} and R is (cf. (3.1))

$$\frac{\Gamma(\frac{n}{2})}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^{m-1} u_i^{\alpha_i-1} \left(1 - \sum_{i=1}^{m-1} u_i \right)^{\alpha_m-1} f(r).$$

Consider the following transformation:

$$z_i = r^2 u_i, \quad i = 1, \dots, m-1,$$

$$z_m = r^2 \left(1 - \sum_{i=1}^{m-1} u_i \right),$$

then

$$u_i = z_i / r^2, \quad i = 1, \dots, m-1,$$

$$r = \left(\sum_{i=1}^m z_i \right)^{1/2}.$$

Thus the Jacobian of the transformation is

$$\text{mod} \left| \begin{array}{cccc} r^{1/2} & & & \\ & r^{1/2} & & \\ & & \ddots & \\ & & & r^{1/2} \\ \frac{1}{2} \left(\sum_{i=1}^m z_i \right)^{-1/2} & \dots & \frac{1}{2} \left(\sum_{i=1}^m z_i \right)^{-1/2} & \end{array} \right| = r^{-2(m-1)} \frac{1}{2} \left(\sum_{i=1}^m z_i \right)^{-1/2} = \frac{1}{2} \left(\sum_{i=1}^m z_i \right)^{-m+1/2}.$$

Now the joint density of z_1, \dots, z_m is

$$(4.2) \quad \frac{\Gamma(\frac{n}{2})}{2 \prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^m z_i^{\alpha_i-1} \left(\sum_{i=1}^m z_i \right)^{-(n-1)/2} f \left(\left(\sum_{i=1}^m z_i \right)^{1/2} \right).$$

If $\underline{x} \sim EC_n(\underline{\mu}, \underline{\Sigma}, \phi)$, then R has a density $f(r)$ if and only if \underline{x} has a density which must have the form

$$|\underline{\Sigma}|^{-1/2} g((\underline{x}-\underline{\mu})' \underline{\Sigma}^{-1} (\underline{x}-\underline{\mu}))$$

for a suitable function $g(\cdot)$. Moreover there exists a relationship between $f(\cdot)$ and $g(\cdot)$, i.e.,

$$f(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} g(r^2).$$

(cf. Cambanis, Huang and Simons (1981)). Substituting the above formula into (4.2), the density of z_1, \dots, z_m becomes

$$(4.3) \quad \frac{\pi^{n/2}}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^m z_i^{\alpha_i-1} g\left(\sum_{i=1}^m z_i\right).$$

In Section 6 we will give some examples of $G_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m; \phi)$. Anderson and Fang (1982) give some further applications of $G_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m; \phi)$.

Theorem 1. Suppose that $\underline{x} \sim EC_n(0, \underline{I}_n, \phi)$, $p(\underline{x}=0) = 0$ and \underline{A} is an $n \times n$ symmetric matrix, then $\underline{x}'\underline{A}\underline{x} \sim G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$ if and only if $\underline{A}^2 = \underline{A}$ and $\text{rk}(\underline{A}) = k$.

Proof. If $\underline{A}^2 = \underline{A}$ and $\text{rk}(\underline{A}) = k$, then there exists an orthogonal matrix $\underline{\Gamma}$ such that $\underline{\Gamma}'\underline{A}\underline{\Gamma} = \begin{pmatrix} \underline{I}_k & 0 \\ 0 & 0 \end{pmatrix}$. Let $\underline{y} = \underline{\Gamma}'\underline{x}$; it can be verified that that $\underline{y} \sim EC_n(0, \underline{I}_n, \phi)$ and $\underline{x}'\underline{A}\underline{x} = \sum_{i=1}^k y_i^2$, where $\underline{y} = (y_1, \dots, y_n)'$. Thus

$$(4.4) \quad \underline{x}'\underline{A}\underline{x} = \sum_{i=1}^k y_i^2 = \|\underline{y}\|^2 \cdot \frac{\sum_{i=1}^k y_i^2}{\|\underline{y}\|^2}.$$

As is known, $\|\underline{y}\|^2$ and $\sum_{i=1}^k y_i^2 / \|\underline{y}\|^2$ are independent and the distribution of $\sum_{i=1}^k y_i^2 / \|\underline{y}\|^2$ does not depend on what the distribution of \underline{y} is in the class of $EC_n(0, \underline{I}_n, \phi)$; therefore we can assume $\underline{y} \sim N_n(0, \underline{I}_n)$. It implies that $\sum_{i=1}^k y_i^2 / \|\underline{y}\|^2 \sim B(\frac{k}{2}, \frac{n-k}{2})$. As $\|\underline{y}\|^2 \stackrel{d}{=} R \sim F(x)$, we have completed the proof of the "if" part.

If $\underline{x}'\underline{A}\underline{x} \sim G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$, write

$$\underline{x}'\underline{A}\underline{x} = \|\underline{x}\|^2 \cdot \frac{\underline{x}'\underline{A}\underline{x}}{\|\underline{x}\|^2} \stackrel{d}{=} R^2 \cdot z,$$

where $R \sim F(x)$, $z \sim B(\frac{k}{2}, \frac{n-k}{2})$ and they are independent. By Lemma 1 we have

$$\frac{\tilde{x}' \tilde{A} \tilde{x}}{\|\tilde{x}\|^2} \stackrel{d}{=} z.$$

From Corollary 1 of Lemma 3 the assertion follows. Q.E.D.

Corollary. Assume that $\tilde{x} \sim EC_n(0, \tilde{\Sigma}, \phi)$ with $\tilde{\Sigma} > 0$ and $p(\tilde{x}=0) = 0$, and \tilde{A} is an $n \times n$ symmetric matrix; then $\tilde{x}' \tilde{A} \tilde{x} \sim G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$ if and only if $\tilde{A} \tilde{\Sigma} \tilde{A} = \tilde{A}$ and $\text{rk}(\tilde{A}) = k$.

Proof. As $\tilde{\Sigma} > 0$, there exists $\tilde{\Sigma}^{\frac{1}{2}} > 0$ such that $\tilde{\Sigma}^{\frac{1}{2}} \tilde{\Sigma}^{\frac{1}{2}} = \tilde{\Sigma}$. Let $\tilde{y} = \tilde{\Sigma}^{-\frac{1}{2}} \tilde{x}$, then $\tilde{y} \sim EC_n(0, \tilde{I}_n, \phi)$. By Theorem 1

$$\tilde{x}' \tilde{A} \tilde{x} = \tilde{y}' (\tilde{\Sigma}^{\frac{1}{2}} \tilde{A} \tilde{\Sigma}^{\frac{1}{2}}) \tilde{y} \sim G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$$

if and only if $(\tilde{\Sigma}^{\frac{1}{2}} \tilde{A} \tilde{\Sigma}^{\frac{1}{2}})^2 = \tilde{\Sigma}^{\frac{1}{2}} \tilde{A} \tilde{\Sigma}^{\frac{1}{2}}$ and $\text{rk}(\tilde{\Sigma}^{\frac{1}{2}} \tilde{A} \tilde{\Sigma}^{\frac{1}{2}}) = k$, i.e. $\tilde{A} \tilde{\Sigma} \tilde{A} = \tilde{A}$ and $\text{rk}(\tilde{A}) = k$. Q.E.D.

It is well-known that $\tilde{A}^2 = \tilde{A}$ if and only if $\text{rk}(\tilde{A}) + \text{rk}(\tilde{I} - \tilde{A}) = n$. Hence $\tilde{A} \tilde{\Sigma} \tilde{A} = \tilde{A}$ if and only if $\text{rk}(\tilde{A}) + \text{rk}(\tilde{\Sigma} - \tilde{A}) = n$. The corollary shows us that $\tilde{\Sigma}$ must be a generalized inverse of \tilde{A} . Kelker (1970) gives a result similar to the corollary, but he assumes finite fourth moments for the components of \tilde{x} .

If $\tilde{x} \sim N_n(0, \tilde{I}_n)$, $\tilde{A}' = \tilde{A}$, $\tilde{A}^2 = \tilde{A}$ and $\text{rk}(\tilde{A}) = k$, then $\tilde{x}' \tilde{A} \tilde{x} \sim \chi_k^2$ by Cochran's Theorem, i.e. $G_2(\frac{k}{2}, \frac{n-k}{2}; \phi) = \chi_k^2$ if $\phi(t) = \exp(-t/2)$. The inverse proposition also holds.

Theorem 2. The distribution $G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$ is χ_k^2 if and only if $\phi(t) = \exp(-t/2)$.

Proof. We only prove the "only if" part. If $z \sim G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$ is χ_k^2 , by the definition of $G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$. We have $z \stackrel{d}{=} R^2 \cdot w$, where $R \sim F(x)$, $w \sim B(\frac{k}{2}, \frac{n-k}{2})$ and R^2 is independent of w . Let $y_1 \sim \chi_k^2$ and $y_0 \sim \chi_n^2$; from (3.10) and the supposition we find

$$\phi_{\ell y_1}(t) = \phi_{\ell z}(t) = \phi_{\ell R^2}(t) \cdot \phi_{\ell w}(t) = \phi_{\ell R^2}(t) \frac{\phi_{\ell y_1}(t)}{\phi_{\ell y_0}(t)},$$

which implies $R^2 \sim \chi_n^2$, i.e. $\phi(t) = \exp(-t/2)$. Q.E.D.

By a method similar to those used in proving Theorem 1 and Theorem 2, we obtain the following theorems and corollaries.

Theorem 3. Suppose that $\underline{x} \sim EC_n(0, \underline{I}_n, \phi)$, $p(\underline{x}=0) = 0$, \underline{A} and \underline{B} are $n \times n$ symmetric matrices, then $(\underline{x}'\underline{A}\underline{x}, \underline{x}'\underline{B}\underline{x}) \sim G_3(\frac{k}{2}, \frac{\ell}{2}, \frac{n-k-\ell}{2}; \phi)$ if and only if $\underline{A}\underline{B} = \underline{0}$, $\underline{A}^2 = \underline{A}$, $\underline{B}^2 = \underline{B}$, $\text{rk}(\underline{A}) = k$ and $\text{rk}(\underline{B}) = \ell$.

Corollary 1. Suppose that $\underline{x} \sim EC_n(0, \underline{\Sigma}, \phi)$ with $\underline{\Sigma} > 0$, $p(\underline{x}=0) = 0$, \underline{A} and \underline{B} are $n \times n$ symmetric matrices, then $(\underline{x}'\underline{A}\underline{x}, \underline{x}'\underline{B}\underline{x}) \sim G_3(\frac{k}{2}, \frac{\ell}{2}, \frac{n-k-\ell}{2}; \phi)$ if and only if $\underline{A}\underline{\Sigma}\underline{A} = \underline{A}$, $\underline{B}\underline{\Sigma}\underline{B} = \underline{B}$, $\text{rk}(\underline{A}) = k$, $\text{rk}(\underline{B}) = \ell$ and $\underline{A}\underline{\Sigma}\underline{B} = \underline{0}$.

Corollary 2. Assume that $\underline{x} \sim EC_n(0, \underline{I}_n, \phi)$, $p(\underline{x}=0) = 0$, \underline{A} and \underline{B} are projection matrices with $\text{rk}(\underline{A}) = k$ and $\text{rk}(\underline{B}) = \ell$ satisfying $\underline{A}\underline{B} = \underline{0}$, then

$$\frac{\ell}{k} \frac{\underline{x}'\underline{A}\underline{x}}{\underline{x}'\underline{B}\underline{x}} \sim F(k, \ell).$$

Proof. As $\underline{A}\underline{B} = \underline{0}$, there exists an orthogonal matrix $\underline{\Gamma}$ such that $\underline{\Gamma}'\underline{A}\underline{\Gamma} = \text{diag}(\underbrace{1, \dots, 1}_k, 0, \dots, 0)$ and $\underline{\Gamma}'\underline{B}\underline{\Gamma} = \text{diag}(\underbrace{0, \dots, 0}_k, \underbrace{1, \dots, 1}_\ell, 0, \dots, 0)$.

Let $\underline{y} = \underline{I}\underline{x}$, then $\underline{y} \sim EC_n(0, \underline{I}, \phi)$, $\underline{x}'\underline{A}\underline{x} = \sum_1^k y_i^2$ and $\underline{x}'\underline{B}\underline{x} = \sum_{k+1}^{k+l} y_i^2$,
thus

$$\frac{\frac{\underline{x}'\underline{A}\underline{x}}{\underline{x}'\underline{B}\underline{x}}}{\frac{\sum_1^k y_i^2}{\sum_{k+1}^{k+l} y_i^2}} = \frac{\frac{\sum_1^k y_i^2}{\sum_{k+1}^{k+l} y_i^2}}{\frac{\sum_1^k y_i^2 / \|\underline{y}\|^2}{\sum_{k+1}^{k+l} y_i^2 / \|\underline{y}\|^2}} \stackrel{d}{=} \frac{\frac{\sum_1^k z_i^2 / \|\underline{z}\|^2}{\sum_{k+1}^{k+l} z_i^2 / \|\underline{z}\|^2}}{\frac{\sum_1^k z_i^2}{\sum_{k+1}^{k+l} z_i^2}} \sim F(k, l),$$

where $\underline{z} \sim N_n(0, \underline{I})$. Q.E.D.

Kelker (1970) treated the distribution of $\sum_1^k \chi_i^2 / \sum_{k+1}^{k+l} \chi_i^2$ by a different method.

Theorem 4. The joint distribution $G_3(\frac{k}{2}, \frac{l}{2}; \frac{n-k-l}{2}; \phi)$ is the product of the distributions of χ_k^2 and χ_l^2 if and only if $\phi(t) = \exp(-t/2)$.

Theorem 5. Suppose that $\underline{x} \sim EC_n(0, \underline{I}, \phi)$, $p(\underline{x}=0) = 0$, \underline{A} and \underline{B} are $n \times n$ symmetric matrices and $(z_1, \dots, z_{n-1}) \sim D_n(\frac{1}{2}, \dots, \frac{1}{2}; \frac{1}{2})$ is independent of R . Then

$$\left(\begin{array}{c} \underline{x}'\underline{A}\underline{x} \\ \underline{x}'\underline{B}\underline{x} \end{array} \right) \stackrel{d}{=} \left(\begin{array}{c} \sum_1^k \lambda_i z_i^2 \\ \sum_{k+1}^{k+l} v_j z_j^2 \end{array} \right)$$

with real numbers λ_i and v_j if and only if $\underline{A}\underline{B} = \underline{0}$ and λ_i 's are the non-zero eigenvalues of \underline{A} , v_j 's are the nonzero eigenvalues of \underline{B} .

The above theorems can be generalized to the case of several quadratic forms.

5. Tripotent Matrices.

A square matrix \underline{A} is said to be tripotent whenever $\underline{A}^3 = \underline{A}$. Anderson and Styan (1980) extended Cochran's Theorem to tripotent matrices. If $\underline{A}^3 = \underline{A}$, the eigenvalues of \underline{A} are 1, -1 and 0. Let p and q denote the number of eigenvalues equal to 1 and -1, respectively. If $\underline{x} \sim N_n(0, \underline{I}_n)$, then $\underline{x}'\underline{A}\underline{x} \stackrel{d}{=} y_1 - y_2$, where y_1 and y_2 are independent, $y_1 \sim \chi_p^2$ and $y_2 \sim \chi_q^2$. The distribution of $y_1 - y_2$ has been given by Press (1969) and Robinson (1965) and we denote it by $H(\frac{p}{2}, \frac{q}{2})$. Similarly, if $(v_1, v_2) \sim D_3(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$ we denote the distribution of $z = v_1 - v_2$ by $H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$.

Lemma 4. Assume that $\underline{x} \sim N_n(0, \underline{I}_n)$ and \underline{A} is an $n \times n$ symmetric matrix. Then $\underline{x}'\underline{A}\underline{x} \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$ if and only if $\underline{A}^3 = \underline{A}$ with p 1's and q -1's as its nonzero eigenvalues.

Proof. Apply Lemma 3 with $g_1(t_1, \dots, t_n) = \sum_1^p t_i - \sum_{p+1}^{p+q} t_j$ and $m = 1$ which completes the proof.

Now we generalize the distribution $H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$ to the case of elliptically contoured distributions. If $z \stackrel{d}{=} v_1 - v_2 \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$, $R \sim F(x)$ and is independent of z , we denote the distribution of $R^2 z$ by $H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2}; \phi)$ where ϕ is related to $F(x)$ as in (1.1). By the same technique used in the proof of Theorem 1 and Lemma 4 we obtain the following theorem.

Theorem 6. Suppose that $\underline{x} \sim EC_n(0, \underline{I}_n, \phi)$, $p(\underline{x}=0) = 0$ and \underline{A} is an $n \times n$ symmetric matrix, then $\underline{x}'\underline{A}\underline{x} \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2}; \phi)$ if and only if $\underline{A}^3 = \underline{A}$ with p 1's and q -1's as its eigenvalues.

Corollary. Suppose that $\underline{x} \sim EC_n(0, \underline{\Sigma}, \phi)$ with $\underline{\Sigma} > 0$, $p(\underline{x}=0) = 0$, and \underline{A} is an $n \times n$ symmetric matrix, then $\underline{x}'\underline{A}\underline{x} \sim H(\frac{p}{2}, \frac{q}{2}, \frac{n-p-q}{2}, \phi)$ if and only if

$$(5.1) \quad \text{rk}(\underline{A}) = \text{rk}(\underline{A} - \underline{A}\underline{\Sigma}\underline{A}) + \text{rk}(\underline{A} + \underline{A}\underline{\Sigma}\underline{A})$$

and $\underline{A}\underline{\Sigma}$ has p 1's and q -1's as its eigenvalues.

Proof. As $\underline{\Sigma} > 0$, there exists $\underline{\Sigma}^{\frac{1}{2}} > 0$ such that $\underline{\Sigma}^{\frac{1}{2}}\underline{\Sigma}^{\frac{1}{2}} = \underline{\Sigma}$. Let $\underline{x} = \underline{\Sigma}^{\frac{1}{2}}\underline{y}$, then $\underline{y} \sim EC_n(0, \underline{I}_n, \phi)$ and $\underline{x}'\underline{A}\underline{x} = \underline{y}'(\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}})\underline{y}$. From Theorem 6 $\underline{x}'\underline{A}\underline{x} \sim H(\frac{p}{2}, \frac{q}{2}, \frac{n-p-q}{2}, \phi)$ if and only if

(i) $\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}}$ has p 1's and q -1's as its eigenvalues, or equivalently $\underline{A}\underline{\Sigma}$ has p 1's and q -1's as its eigenvalues.

(ii) $(\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}})^3 = (\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}})$. Anderson and Styan (1980) point out that $\underline{B}^3 = \underline{B}$ if and only if $\text{rk}(\underline{B}) = \text{rk}(\underline{B} + \underline{B}^2) + \text{rk}(\underline{B} - \underline{B}^2)$ for any square matrix \underline{B} . Thus $(\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}})^3 = \underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}}$ if and only if

$$\begin{aligned} \text{rk}(\underline{A}) &= \text{rk}(\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}}) = \text{rk}(\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}} + \underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}\underline{A}\underline{\Sigma}^{\frac{1}{2}}) + \text{rk}(\underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}^{\frac{1}{2}} - \underline{\Sigma}^{\frac{1}{2}}\underline{A}\underline{\Sigma}\underline{A}\underline{\Sigma}^{\frac{1}{2}}) \\ &= \text{rk}(\underline{A} + \underline{A}\underline{\Sigma}\underline{A}) + \text{rk}(\underline{A} - \underline{A}\underline{\Sigma}\underline{A}) . \quad \text{Q.E.D.} \end{aligned}$$

6. Applications.

In this section some applications are given. In the first part we consider the theory on linear regression with the error being distributed according to an elliptically contoured distribution. We will find the distribution of the sample variance, the distribution of the ratio of the sample mean to the sample standard deviation in the second part. In addition

we will give the distributions of quadratic forms for the multivariate t-distribution, the multivariate uniform distribution, etc.

6.1. Linear Regression Analysis. We consider the following model:

$$(6.1) \quad \begin{cases} \underline{y}_{n \times 1} = \underline{X}_{n \times p} \underline{\beta}_{p \times 1} + \underline{e}_{n \times 1}, & \text{rk}(\underline{X}) = p < n \\ E(\underline{e}) = \underline{0}, & E(\underline{e}\underline{e}') = \underline{\Sigma} > 0 \\ \underline{e} \sim EC_n(\underline{0}, \underline{\Sigma}, \phi) . \end{cases}$$

There exists a $p \times p$ matrix \underline{A} such that $\underline{A}'\underline{A} = \underline{\Sigma}$ and

$$(6.2) \quad \underline{e} \stackrel{d}{=} \underline{R}\underline{A}'\underline{u}^{(n)} .$$

Hence

$$(6.3) \quad \underline{y} \stackrel{d}{=} \underline{X}\underline{\beta} + \underline{R}\underline{A}'\underline{u}^{(n)} ,$$

i.e., $\underline{y} \sim EC_n(\underline{X}\underline{\beta}, \underline{\Sigma}, \phi)$.

Minimizing $\underline{e}'\underline{e} = (\underline{y} - \underline{X}\underline{\beta})'(\underline{y} - \underline{X}\underline{\beta})$ with respect to $\underline{\beta}$ gives the least squares estimator

$$(6.4) \quad \hat{\underline{\beta}} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{y} .$$

In order to find the distribution of $\hat{\underline{\beta}}$ we need the following lemma.

Lemma 5. Assume $\underline{y} \sim EC_n(\underline{\mu}, \underline{I}_n, \phi)$ and \underline{A} is $n \times p$ matrix with $\text{rk}(\underline{A}) = p < n$; then $\underline{X} \equiv \underline{A}'\underline{y} \sim EC_p(\underline{A}'\underline{\mu}, \underline{A}'\underline{A}, \phi)$, where $\phi \in \Phi_p$ corresponds to R^* and

(6.5)

$$R^* \stackrel{d}{=} R \cdot b ,$$

where $b \geq 0$, $b^2 \sim B(\frac{1}{2}p, \frac{1}{2}(n-p))$, and b is independent of R .

Proof. Consider the singular value decomposition of \underline{A} , i.e., there exist orthogonal matrices $\underline{\Gamma}_{p \times p}$ and $\underline{\Delta}_{n \times n}$ such that

$$\underline{A}' = \underline{\Gamma}(\underline{D}_r \ 0)\underline{\Delta}' ,$$

where $\underline{D}_r = \text{diag}(\lambda_1, \dots, \lambda_p)$ and $\lambda_1, \dots, \lambda_p$ are the roots of $|\underline{A}'\underline{A} - \lambda \underline{I}| = 0$.

From the assumption

$$\underline{y} \stackrel{d}{=} \underline{\mu} + R\underline{u}^{(n)} ,$$

thus

$$\begin{aligned} \underline{x} &\stackrel{d}{=} \underline{A}'\underline{\mu} + R\underline{A}'\underline{u}^{(n)} \\ &= \underline{A}'\underline{\mu} + R\underline{\Gamma}(\underline{D}_\lambda \ 0)\underline{\Delta}'\underline{u}^{(n)} \\ &\stackrel{d}{=} \underline{A}'\underline{\mu} + R\underline{\Gamma}(\underline{D}_\lambda \ 0)\underline{u}^{(n)} , \end{aligned}$$

because $\underline{u}^{(n)} \stackrel{d}{=} \underline{\Delta}'\underline{u}^{(n)}$. Let $\underline{z} \sim N_n(0, \underline{I}_n)$ and

$$\underline{u}^{(n)} = \begin{pmatrix} \underline{u}(1) \\ \underline{u}(2) \end{pmatrix} , \quad \underline{z} = \begin{pmatrix} \underline{z}(1) \\ \underline{z}(2) \end{pmatrix} ,$$

where $\underline{u}(1)$ and $\underline{z}(1)$ are $p \times 1$ vectors. Based on the relationship between $\underline{u}^{(n)}$ and \underline{z} we have

$$\begin{aligned} \underline{x} &\stackrel{d}{=} \underline{A}'\underline{\mu} + R\underline{\Gamma}\underline{D}_\lambda \underline{u}(1) \stackrel{d}{=} \underline{A}'\underline{\mu} + R\underline{\Gamma}\underline{D}_\lambda \underline{z}(1) / \|\underline{z}\| \\ &= \underline{A}'\underline{\mu} + R\underline{\Gamma}\underline{D}_\lambda (\|\underline{z}(1)\| / \|\underline{z}\|) \underline{z}(1) / \|\underline{z}(1)\| \\ &\stackrel{d}{=} \underline{A}'\underline{\mu} + (Rb)(\underline{\Gamma}\underline{D}_\lambda)\underline{u}^{(p)} . \end{aligned}$$

Since $\underline{\underline{A}}'\underline{\underline{A}} = (\underline{\underline{\Gamma D}}_\lambda)(\underline{\underline{\Gamma D}}_\lambda)'$, the Lemma follows. Q.E.D.

By Lemma 5 and (6.4) we have

$$(6.6) \quad \begin{aligned} \hat{\underline{\underline{\beta}}} &\stackrel{d}{=} (\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}'[\underline{\underline{X}}\underline{\underline{\beta}} + \underline{\underline{R}}\underline{\underline{A}}'\underline{\underline{u}}^{(n)}] \\ &= \underline{\underline{\beta}} + \underline{\underline{R}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}'\underline{\underline{A}}'\underline{\underline{u}}^{(n)}, \end{aligned}$$

or
$$\hat{\underline{\underline{\beta}}} \sim EC_p(\underline{\underline{\beta}}, (\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}'\underline{\underline{\Sigma}}\underline{\underline{X}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}, \phi),$$

where ϕ is defined by Lemma 5. In particular, if $\underline{\underline{\Sigma}} = \sigma^2 \underline{\underline{I}}_p$, then

$$\hat{\underline{\underline{\beta}}} \sim EC_p(\underline{\underline{\beta}}, \sigma^2(\underline{\underline{X}}'\underline{\underline{X}})^{-1}, \phi).$$

Denote

$$(6.7) \quad \begin{aligned} s &= (\underline{\underline{y}} - \underline{\underline{X}}\hat{\underline{\underline{\beta}}})'(\underline{\underline{y}} - \underline{\underline{X}}\hat{\underline{\underline{\beta}}}) \\ &= \underline{\underline{y}}'[\underline{\underline{I}} - \underline{\underline{X}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}']\underline{\underline{y}} \\ &= \underline{\underline{e}}'[\underline{\underline{I}} - \underline{\underline{X}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}']\underline{\underline{e}}. \end{aligned}$$

If $\underline{\underline{e}} \sim EC_n(0, \sigma^2 \underline{\underline{I}}_n, \phi)$, $p(\underline{\underline{e}}=0) = 0$ and $\text{rk}\underline{\underline{X}} = p$, then $s \sim \sigma^2 G_2(\frac{n-p}{2}; \frac{p}{2}; \phi)$ by Theorem 1. We summarize the above results in the following theorem.

Theorem 7. Assume $\underline{\underline{e}} \sim EC_n(0, \sigma^2 \underline{\underline{I}}_n, \phi)$, $p(\underline{\underline{e}}=0) = 0$ and $\text{rk}\underline{\underline{X}} = p < n$, then

$$\hat{\underline{\underline{\beta}}} \sim EC_p(\underline{\underline{\beta}}, \sigma^2(\underline{\underline{X}}'\underline{\underline{X}})^{-1}, \phi)$$

and

$$s \sim \sigma^2 G_2(\frac{n-p}{2}; \frac{p}{2}; \phi).$$

Now we consider the linear hypothesis

$$(6.8) \quad H: \underline{\underline{H}}\underline{\underline{\beta}} = \underline{\underline{c}},$$

where

$$\underline{\underline{H}}: q \times p, \text{rk}(\underline{\underline{H}}) = q < p.$$

Under the condition of $\underline{\underline{H}}\underline{\underline{\beta}} = \underline{\underline{c}}$ minimizing $\underline{\underline{e}}'\underline{\underline{e}}$ with respect to $\underline{\underline{\beta}}$ gives the least squares estimator

$$(6.9) \quad \underline{\underline{\beta}}_H = \underline{\underline{\hat{\beta}}} - (\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{H}}'(\underline{\underline{H}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}(\underline{\underline{H}}\underline{\underline{\hat{\beta}}} - \underline{\underline{c}})),$$

where $\underline{\underline{\beta}}$ is given by (6.4), and the corresponding $(\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}}_H)'(\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}}_H)$ becomes

$$(6.10) \quad \begin{aligned} s_H &= (\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}}_H)'(\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}}_H) = (\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}} + \underline{\underline{X}}\underline{\underline{\hat{\beta}}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}}_H)'(\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}} + \underline{\underline{X}}\underline{\underline{\hat{\beta}}} - \underline{\underline{X}}\underline{\underline{\hat{\beta}}}_H) \\ &= s + (\underline{\underline{H}}\underline{\underline{\hat{\beta}}} - \underline{\underline{c}})'(\underline{\underline{H}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{H}}')^{-1}(\underline{\underline{H}}\underline{\underline{\hat{\beta}}} - \underline{\underline{c}}), \end{aligned}$$

where s is defined by (6.7).

The statistics testing the hypothesis (6.8) is

$$(6.11) \quad \lambda = \frac{s_H - s}{s}.$$

When the hypothesis is true we have $\underline{\underline{c}} = \underline{\underline{H}}\underline{\underline{\beta}}$ and

$$(6.12) \quad (\underline{\underline{H}}\underline{\underline{\hat{\beta}}} - \underline{\underline{c}}) = \underline{\underline{H}}(\underline{\underline{\hat{\beta}}} - \underline{\underline{\beta}}) \stackrel{d}{=} R\underline{\underline{H}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}'\underline{\underline{A}}'\underline{\underline{u}}^{(n)} \stackrel{d}{=} \underline{\underline{H}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}'\underline{\underline{e}}.$$

Thus

$$(6.13) \quad \begin{aligned} s_H - s &= \underline{\underline{e}}'[\underline{\underline{X}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{H}}'(\underline{\underline{H}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{H}}')^{-1}\underline{\underline{H}}(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}']\underline{\underline{e}} \\ &\equiv \underline{\underline{e}}'\underline{\underline{D}}_H\underline{\underline{e}} \quad (\text{say}). \end{aligned}$$

It is easy to see that $\underline{D}_H^2 = \underline{D}_H$, $\text{rk} \underline{D}_H = q$, and $\underline{D}_H(\underline{I} - \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}') = \underline{0}$. From (6.13) and (6.7) λ can be expressed as

$$(6.14) \quad \lambda = \frac{\underline{e}' \underline{D}_H \underline{e}}{\underline{e}' [\underline{I} - \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'] \underline{e}},$$

and

$$\frac{n-p}{q} \lambda \sim F(q, n-p).$$

(cf. Corollary 2 of Theorem 3).

Theorem 8. Assume $\underline{e} \sim EC_n(0, \sigma^2 \underline{I}_n, \phi)$, $p(\underline{e}=0) = 0$ and $\text{rk} \underline{X} = p$, when the hypothesis (6.8) is true the distribution of $(n-p)\lambda/q$ is $F(q, n-p)$.

6.2. Some examples.

Example 1. If $\underline{x} \sim EC_n(0, \underline{I}_n, \phi)$ and $p(\underline{x}=0) = 0$, let

$$(6.15) \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{and} \quad s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2.$$

We can express $s^2 = \underline{x}'(\underline{I}_n - \underline{M})\underline{x}$ with $\underline{M} = \frac{1}{n} \underline{e}_n \underline{e}_n'$. It is easy to check that $(\underline{I}_n - \underline{M})^2 = \underline{I}_n - \underline{M}$ and $\text{rk}(\underline{I}_n - \underline{M}) = n-1$. By Theorems 1 and 2 $s^2 \sim G_2(\frac{n-1}{2}; \frac{1}{2}; \phi)$ and $s^2 \sim \chi_{n-1}^2$ if and only if $\phi(t) = \exp(-t/2)$, i.e. $\underline{x} \sim N_n(\underline{0}, \underline{I}_n)$.

Further, if $\underline{x} \sim EC_n(0, \underline{\Sigma}, \phi)$ with $\underline{\Sigma} > 0$ and $p(\underline{x}=0) = 0$, and \bar{x} and s^2 are defined by (6.15), by the corollary of Theorem 1 $s^2 \sim G_2(\frac{n-1}{2}; \frac{1}{2}; \phi)$ if and only if $\underline{\Sigma}$ is a generalized inverse of $(\underline{I}_n - \underline{M})$. Let $\underline{\Gamma}$ be an orthogonal matrix with the first row $(1/\sqrt{n}, \dots, 1/\sqrt{n})$, then

$$\underline{\Gamma}'(\underline{I}_n - \underline{M})\underline{\Gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \underline{I}_{n-1} \end{pmatrix}.$$

or

$$\tilde{I}-\tilde{M} = \tilde{\Gamma} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{I}_{n-1} \end{pmatrix} \tilde{\Gamma}' .$$

It can be verified that any generalized inverse $(\tilde{I}-\tilde{M})^-$ of $(\tilde{I}-\tilde{M})$ can be written as

$$(\tilde{I}-\tilde{M})^- = \tilde{\Gamma} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & \tilde{I}_{n-1} \end{pmatrix} \tilde{\Gamma}' ,$$

where c_{11} , c_{12} and c_{21} are arbitrary. Hence $\tilde{\Sigma}$ must have the above structure, but here $\tilde{\Sigma} > 0$, $\{c_{ij}\}$ have to satisfy the conditions: $c_{21} = c_{12}'$ and $c_{11} > c_{12}c_{21}$.

Example 2. Assume $\tilde{x} \sim EC_n(0, \tilde{I}_n, \phi)$, $p(\tilde{x}=0) = 0$, and \bar{x} and s^2 are defined by (6.15), we want to point out that

$$(6.16) \quad t = \sqrt{n(n-1)} \frac{\bar{x}}{s} \sim t_{(n-1)} .$$

Let $\tilde{\Gamma}$ be the same as Example 1 and $\tilde{z} = \tilde{\Gamma}\tilde{x}$, then $\tilde{y} \sim EC_n(0, \tilde{I}_n, \phi)$ and $y_1 = \sqrt{n} \bar{x}$, $s^2 = \sum_2^n y_i^2$. If $\tilde{z} \sim N_n(0, \tilde{I}_n)$, then

$$t = \frac{\sqrt{n-1} y_1}{\sqrt{\sum_2^n y_i^2}} = \sqrt{n-1} \frac{y_1/\|\tilde{y}\|}{\sqrt{\sum_2^n y_i^2/\|\tilde{y}\|^2}} \stackrel{d}{=} \sqrt{n-1} \frac{z_1/\|\tilde{z}\|}{\sqrt{\sum_2^n z_i^2/\|\tilde{z}\|^2}} = \sqrt{n-1} \frac{z_1}{\sqrt{\sum_2^n z_i^2}} \sim t_{(n-1)} ,$$

because $(y_1/\|\tilde{y}\|, \dots, y_n/\|\tilde{y}\|) \stackrel{d}{=} (z_1/\|\tilde{z}\|, \dots, z_n/\|\tilde{z}\|)$, hence their Borel functions have the same distribution.

Example 3. Multivariate t-distribution. If $\underline{x} \sim N_n(\underline{\mu}, \underline{\Sigma})$, $s \sim \chi_\nu$ (i.e., $s^2 \sim \chi_\nu^2$) and \underline{x} is independent of s , the distribution of $\underline{y} \equiv \sqrt{\nu} \underline{x}/s$ is called the multivariate t-distribution. (cf. Johnson and Kotz (1972)). When $\underline{\mu} = \underline{0}$, $\underline{\Sigma} = \underline{I}_n$, the density of \underline{y} is

$$(6.17) \quad p_{\underline{y}}(\underline{y}) = \frac{\Gamma(\frac{1}{2}(n+\nu))}{(\pi\nu)^{\frac{1}{2}n} \Gamma(\frac{1}{2}\nu)} (1+\nu^{-1} \sum_{i=1}^n y_i^2)^{-\frac{1}{2}(n+\nu)}.$$

It can be found that the density of $\|\underline{y}\|$ is

$$(6.18) \quad p_{\|\underline{y}\|}(u) = \frac{2 \Gamma(\frac{1}{2}(n+\nu))}{\Gamma(\frac{1}{2}n) \Gamma(\frac{1}{2}\nu) \nu^{\frac{1}{2}n}} u^{n-1} (1+\nu^{-1} u^2)^{-\frac{1}{2}(n+\nu)}, \quad u \geq 0.$$

Obviously the multivariate t-distribution is an elliptically contoured distribution. If $\underline{x} \sim N_n(\underline{\mu}, \underline{\Sigma})$, from (1.2)

$$\underline{x} \stackrel{d}{=} \underline{\mu} + R \underline{A}' \underline{u}^{(n)}$$

where $R \sim \chi_n$ is independent of $\underline{u}^{(n)}$ and $\underline{A}'\underline{A} = \underline{\Sigma}$. Thus if $\underline{\mu} = \underline{0}$

$$(6.19) \quad \underline{y} \stackrel{d}{=} (\sqrt{\nu} \frac{R}{s}) \underline{A}' \underline{u}^{(n)},$$

which is the stochastic representation of \underline{y} . The density (6.18) of $\sqrt{\nu} R/s$ can be motivated by the F-distribution because $\frac{1}{n} (\sqrt{\nu} \frac{R}{s})^2 = \frac{\nu}{n} \frac{R^2}{s^2} \sim F(n, \nu)$.

If \underline{A} is an $n \times n$ symmetric matrix,

$$\underline{y}' \underline{A} \underline{y} = \nu \frac{\underline{x}' \underline{A} \underline{x}}{s^2}.$$

As $\underline{x}' \underline{A} \underline{x}$ is independent of s^2 and $\underline{x}' \underline{A} \underline{x} \sim \chi_k^2$ if and only if $\underline{A} \underline{\Sigma} \underline{A} = \underline{A}$ and $\text{rk}(\underline{A}) = k$, hence $\frac{1}{k} \underline{y}' \underline{A} \underline{y} \sim F(k, \nu)$ if and only if $\underline{A} \underline{\Sigma} \underline{A} = \underline{A}$ and $\text{rk}(\underline{A}) = k$.

Further if $\underline{\Sigma} = \underline{I}_n$, \underline{A} and \underline{B} are projection matrices with $\text{rk}(\underline{A}) = k$, $\text{rk}(\underline{B}) = l$, then $\frac{1}{v} (\underline{y}'\underline{A}\underline{y}, \underline{y}'\underline{B}\underline{y})$ is distributed according to multivariate inverted Dirichlet distribution (cf. p. 238, Johnson and Kotz (1972)).

When $v = 1$ the multivariate t-distribution reduces to multivariate Cauchy distribution.

Example 4. Uniform distribution in the unit sphere in \mathbb{R}^n . If \underline{x} is distributed according to the uniform distribution in the unit sphere in \mathbb{R}^n , its density is

$$(6.20) \quad p_{\underline{x}}(\underline{x}) = \begin{cases} \frac{\Gamma(\frac{n+2}{2})}{\pi^{n/2}} & \text{if } \sum_{i=1}^n x_i^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that the density of $\|\underline{x}\|$ is

$$p_{\|\underline{x}\|}(u) = nu^{n-1} \quad \text{for } 0 \leq u \leq 1,$$

and the density of $\|\underline{x}\|^2$ is

$$p_{\|\underline{x}\|^2}(u) = \frac{1}{2} nu^{\frac{1}{2}n-1} \quad \text{for } 0 \leq u \leq 1.$$

In this case the density of $G_2(\frac{k}{2}; \frac{n-k}{2}; \phi)$ is (cf. (4.1))

$$\begin{aligned}
& \frac{n \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} x^{\frac{k}{2}-1} \int_{\sqrt{x}}^1 r^{-(n-2)} (r^2-x)^{\frac{n-k}{2}-1} r^{n-1} dr \\
&= \frac{n \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} x^{\frac{k}{2}-1} \frac{1}{n-1} (r^2-x)^{\frac{n-k}{2}} \Big|_{\sqrt{x}}^1 \\
&= \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n+2-k}{2})} x^{\frac{k}{2}-1} (1-x)^{\frac{n+2-k}{2}-1}, \quad 0 \leq x \leq 1.
\end{aligned}$$

It is the density of $B(\frac{k}{2}, \frac{n+2-k}{2})$. Similarly, $G_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_m; \phi)$ is equal to $D_m(\alpha_1, \dots, \alpha_{m-1}; \alpha_{m+1})$, where $2(\alpha_1 + \dots + \alpha_m) = n$.

By the theorems in Section 4 we obtain the following conclusions: If \tilde{A} is an $n \times n$ symmetric matrix, then $\tilde{x}' \tilde{A} \tilde{x} \sim B(\frac{k}{2}, \frac{n+2-k}{2})$ if and only if $\tilde{A}^2 = \tilde{A}$ and $\text{rk}(\tilde{A}) = k$. If $\tilde{A}_i, i = 1, \dots, m-1$, are $n \times n$ symmetric matrices, then

$$(\tilde{x}' \tilde{A}_1 \tilde{x}, \dots, \tilde{x}' \tilde{A}_{m-1} \tilde{x}) \sim D_m(\frac{\alpha_1}{2}, \dots, \frac{\alpha_{m-1}}{2}; \frac{n+2-\alpha_1-\dots-\alpha_{m-1}}{2})$$

if and only if $\tilde{A}_i^2 = \tilde{A}_i, \text{rk}(\tilde{A}_i) = \alpha_i, i = 1, \dots, m-1$ and $\tilde{A}_i \tilde{A}_j = 0$ for $i \neq j$.

Clearly, we can extend the above conclusions to the uniform distribution in ellipsoid.

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